Residual Ratio Tracking for Estimating Attenuation in Participating Media (supplementary material)

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1 Introduction

In this supplementary material, we provide complete algorithmic definitions of the *ratio* and *residual ratio* tracking estimators. We also prove the correctness of both algorithms and include additional results that did not fit in the paper.

2 Definition of Ratio Tracking

In order to ease the comparison to delta tracking, we adapt some of the definitions by Coleman [1968] who presented a mathematically rigorous description of the delta tracking technique. Our definition of ratio tracking follows a similar path.

We are interested in estimating transmittance $T(d)$ along a straight line up to a certain distance d, i.e. evaluating the following equation:

$$T(d) = \exp\left(-\int_0^d \mu(x)\mathrm{d}x\right).$$ (1)

Let:

- $(S_1, S_2, ...S_n, ...)$ denote an infinite sequence of independent random variables having a common distribution.

$$P(S_i \le s) = F_S(s) = \int_0^s \bar{\mu}\exp\left(-\bar{\mu}x\right)\mathrm{d}x,$$ (2)

$$p_S(x) = \mathrm{d}F_S(x) = \bar{\mu}\exp\left(-\bar{\mu}x\right),$$ (3)

  where $s \ge 0$. The random variable S_i represents a free path length in a homogeneous medium with extinction coefficient $\bar{\mu}$.

- $(C_1, C_2, ...C_n, ...)$ denotes a sequence of random variables that represent the cumulative sums of sub-steps:

$$C_0 = S_0 = 0,$$ (4)

$$C_i = \sum_{j=1}^i S_j = C_{i-1} + S_i.$$ (5)

- $\kappa(x) = \mu(x)/\bar{\mu}$ and $\iota(x) = 1 - \kappa(x)$ are the local ratios of real and fictitious particles w.r.t $\bar{\mu}$, respectively.

- K is a random variable with realizations k denoting the *maximum* number i for which $C_i \le d$. K represents the number of free path length samples that the tracking performs before reaching d.

Finally, T denotes a random variable that estimates transmittance $T(d)$ as:

$$T = \prod_{i=1}^K \iota(C_i).$$ (6)

We also recognize a set of mutually exclusive random variables $\{T_1, T_2, \cdots T_k, \cdots\}$; T_k represents realizations of T where K takes on a specific value k:

$$T_k = \prod_{i=1}^k \iota(C_i) = \prod_{i=1}^k \left(1 - \frac{\mu(C_i)}{\bar{\mu}}\right).$$ (7)

3 Proof of Ratio Tracking

Coleman [1968] demonstrated the unbiasedness of delta tracking by showing that it generates free flight distances from density $\mu(x)\exp\left(-\int_0^d \mu(x)\mathrm{d}x\right)$. Our goal here is slightly different: we want to show that T has the following expected value:

$$E[T] = \exp\left(-\int_0^d \mu(x)\mathrm{d}x\right) = T(d).$$ (8)

Denoting τ_i realizations of T_i, the expected value of T can then be written as:

$$E[T] = E\left[\sum_{i=0}^\infty T_i\right] = \sum_{i=0}^\infty E[T_i] = \sum_{i=0}^\infty \int_{-\infty}^\infty \tau_i \mathrm{d}P(\tau_i).$$ (9)

In order to gain some insight, we first express the expected value of T_0 and T_1. T_0 represents all realizations with the first tentative free flight distance S_1 already exceeding d. Since the value of T_0 equals to 1 (see Equation (7)), the expected value reduces to the probability of S_1 taking on values greater than d:

$$E[T_0] = P(S_1 > d) = \int_d^\infty p_S(x)\mathrm{d}x = \exp\left(-\bar{\mu}d\right).$$ (10)

In the case of T_1, which represents events where $S_1 \le d$ and $S_2 > d - S_1$, Equation (7) evaluates to $\iota(x)$. The expectancy of T_1 reads:

$$\begin{aligned}
E[T_1] &= \int_0^d \iota(x)P(S_2 > d - x)\mathrm{d}P(S_1 \le x) \\
&= \int_0^d \iota(x_1)p_S(x_1)\int_{d-x_1}^\infty p_S(x_2)\mathrm{d}x_2\mathrm{d}x_1 \\
&= \int_0^d \iota(x_1)\bar{\mu}\exp\left(-\bar{\mu}x_1\right)\exp\left(-\bar{\mu}(d - x_1)\right)\mathrm{d}x_1 \\
&= \bar{\mu}\exp\left(-\bar{\mu}d\right)\int_0^d \iota(x)\mathrm{d}x.
\end{aligned}$$ (11)

We can analogously express the expected value of T_k, which represents realizations that satisfy $C_k \leq d$ and $C_{k+1} > d - C_k$, as:

$$\mathrm{E}[T_k] = \int_0^d \iota(x_1) p_S(x_1) \int_0^{d-x_1} \iota(x_1 + x_2) p_S(x_2) \cdots$$

$$\cdots \int_0^{d-\sum_{j=1}^{k-1} x_j} \iota\Big(\sum_{j=1}^k x_j\Big) p_S(x_k)$$

$$\times \int_{d-\sum_{j=1}^k x_j}^{\infty} p_S(x_{k+1}) \mathrm{d}x_{k+1} \mathrm{d}x_k \cdots \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \int_0^d \iota(x_1) \int_0^{d-x_1} \iota(x_1 + x_2) \cdots \int_0^{d-\sum_{j=1}^{k-1} x_j} \iota\Big(\sum_{j=1}^k x_j\Big)$$

$$\times \int_{d-\sum_{j=1}^k x_j}^{\infty} \bar{\mu}^k \exp\left(-\bar{\mu}\Big(x_1 + \cdots + x_k + \big(d - \sum_{j=1}^k x_j\big)\Big)\right)$$

$$\mathrm{d}x_{k+1} \mathrm{d}x_k \cdots \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \bar{\mu}^k \exp\left(-\bar{\mu}d\right) \int_0^d \iota(x_1) \int_0^{d-x_1} \iota(x_1 + x_2) \cdots$$

$$\cdots \int_0^{d-\sum_{j=1}^{k-1} x_j} \iota\Big(\sum_{j=1}^k x_j\Big) \mathrm{d}x_k \cdots \mathrm{d}x_2 \mathrm{d}x_1. \quad (12)$$

Similarly to Coleman, we substitute z_i for $\sum_{j=0}^i x_j$ allowing to write the upper bounds and the arguments of ι succinctly as:

$$\mathrm{E}[T_k] = \bar{\mu}^k \exp\left(-\bar{\mu}d\right) \int_0^d \iota(z_1) \int_0^{d-z_1} \iota(z_2) \cdots$$

$$\cdots \int_0^{d-z_{k-1}} \iota(z_k) \mathrm{d}z_k \cdots \mathrm{d}z_2 \mathrm{d}z_1. \quad (13)$$

The multiple integrals integrate $\prod_{j=1}^k \iota(z_j)$ over a k-dimensional simplex, which can be written concisely as:

$$\mathrm{E}[T_k] = \bar{\mu}^k \exp\left(-\bar{\mu}d\right) \frac{\left(\int_0^d \iota(x) \mathrm{d}x\right)^k}{k!}. \quad (14)$$

Finally, we express the expected value of T yielding Equation (8):

$$\mathrm{E}[T] = \sum_{k=0}^{\infty} \mathrm{E}[T_k]$$

$$= \sum_{k=0}^{\infty} \bar{\mu}^k \exp\left(-\bar{\mu}d\right) \frac{\left(\int_0^d \iota(x) \mathrm{d}x\right)^k}{k!}$$

$$= \exp\left(-\bar{\mu}d\right) \sum_{i=0}^{\infty} \frac{\left(\bar{\mu} \int_0^d \iota(x) \mathrm{d}x\right)^k}{k!}$$

$$= \exp\left(-\bar{\mu}d\right) \exp\left(\bar{\mu} \int_0^d 1 - \frac{\mu(x)}{\bar{\mu}} \mathrm{d}x\right)$$

$$= \exp\left(-\int_0^d \bar{\mu} \mathrm{d}x\right) \exp\left(\int_0^d \bar{\mu} - \mu(x) \mathrm{d}x\right)$$

$$= \exp\left(-\int_0^d \mu(x) \mathrm{d}x\right) \quad (15)$$

4 Definition of Residual Ratio Tracing

For brevity, we only point out the differences to ratio tracking. The proof of correctness follows in Section 5. Let:

- random variables S_n and C_n be defined analogously to ratio tracking (except for the value of $\bar{\mu}_r$).

- $\kappa(x) = \frac{\mu(x) - \mu_c}{\bar{\mu}_r}$ and $\iota(x) = 1 - \kappa(x)$. Note that $\kappa(x)$ and $\iota(x)$ can no longer be interpreted as local ratios, as their values can be arbitrary and only need to add up to 1.

- K, T, and T_k are defined analogously to ratio tracking, except for the values of $\kappa(x)$. T and T_k represent random variables that estimate the residual transmittance.

5 Proof of Residual Ratio Tracking

We want to show that T has the following expected value:

$$\mathrm{E}[T] = \exp\left(-\int_0^d \mu(x) - \mu_c \mathrm{d}x\right) = \frac{T(d)}{\exp\left(-\mu_c d\right)}. \quad (16)$$

The proof is very similar to the proof of ratio tracking. Indeed, all the differences between ratio and residual ratio tracking are hidden in the definition of $\bar{\mu}_r$ and $\kappa(x)$. We can thus directly apply Equations (9) to (14) leading to:

$$\mathrm{E}[T] = \exp\left(-\bar{\mu}_r d\right) \sum_{k=0}^{\infty} \frac{\left(\bar{\mu}_r \int_0^d \iota(x) \mathrm{d}x\right)^k}{k!}$$

$$= \exp\left(-\bar{\mu}_r d\right) \exp\left(\bar{\mu}_r \int_0^d 1 - \frac{\mu(x) - \mu_c}{\bar{\mu}_r} \mathrm{d}x\right)$$

$$= \exp\left(-\int_0^d \bar{\mu}_r\right) \exp\left(\int_0^d \bar{\mu}_r - (\mu(x) - \mu_c) \mathrm{d}x\right)$$

$$= \exp\left(-\int_0^d \mu(x) - \mu_c \mathrm{d}x\right) \quad (17)$$

6 Additional Results

Figure 1 and Figure 2 show additional illustrations of the residual ratio tracking with different extinction functions.

Figure 3 shows additional visualizations of the variance, cost, and the effective variance, and how these depend on different values of μ_c with different extinction functions.

Figures 4 and 5 show full renderings of the insets used in the teaser of the paper. In Figure 6, we present a convergence plot that shows how the RMSE evolves with increased number of $\mu(x)$ evaluations.

References

COLEMAN, W. A. 1968. Mathematical verification of a certain Monte Carlo sampling technique and applications of the technique to radiation transport problems. *Nuclear Science and Engineering 32*, 1 (Apr.), 76–81.
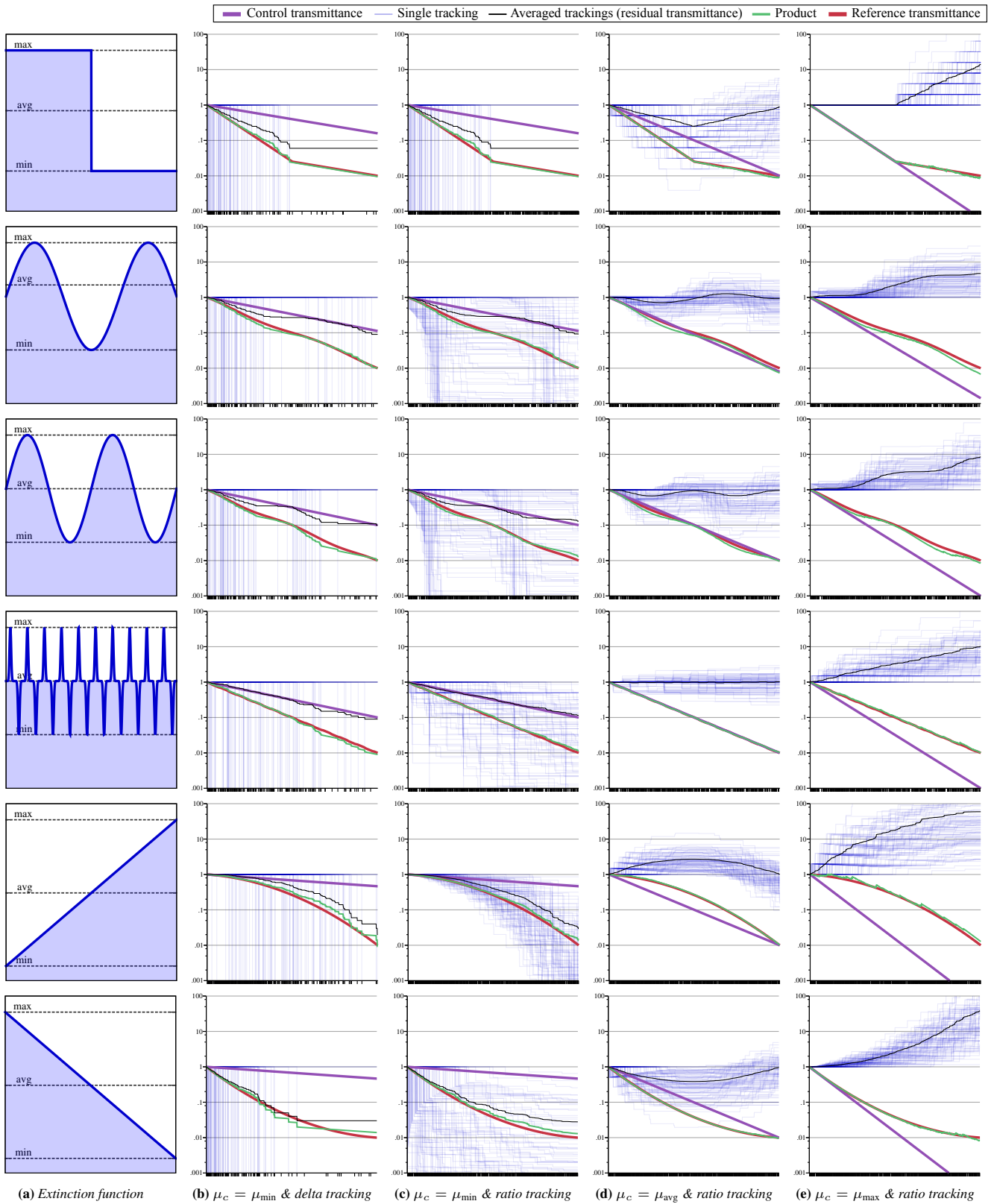
Figure 1: *Comparison of residual tracking with different control extinction coefficients. In (**b**) and (**c**), we analytically compute the control transmittance (purple) based on the minimum $\mu(x)$ along the ray and then apply delta tracking (**b**) and ratio tracking (**c**) to numerically solve the transmittance (blue curves - individual trackings, black curve - average) through the residual medium. The product of the control and the residual transmittance is represented by the green curve. Ratio tracking can be used with arbitrary control extinctions: in (**d**) and (**e**), we show examples with the average and maximum $\mu(x)$ used as the control extinction.*
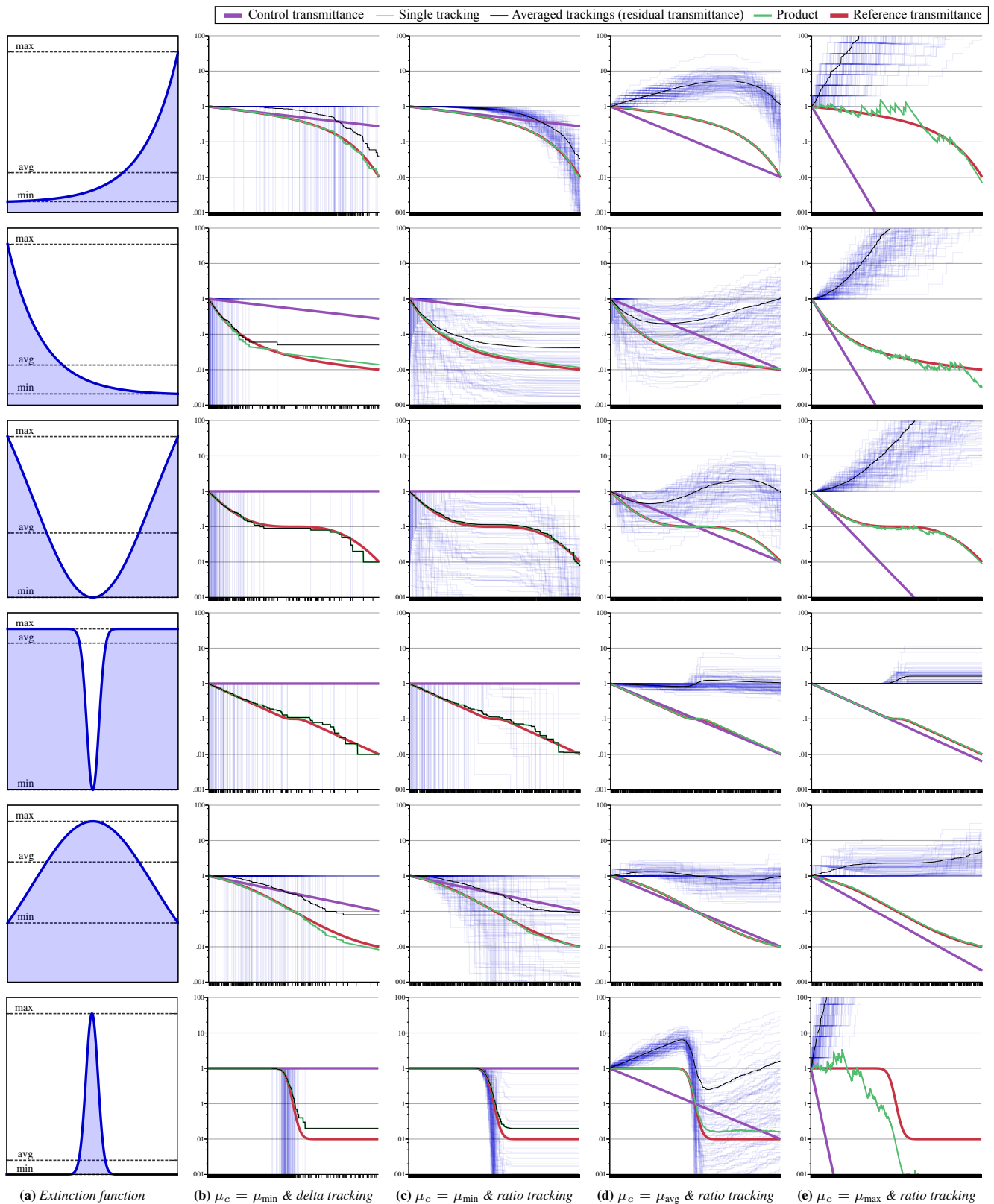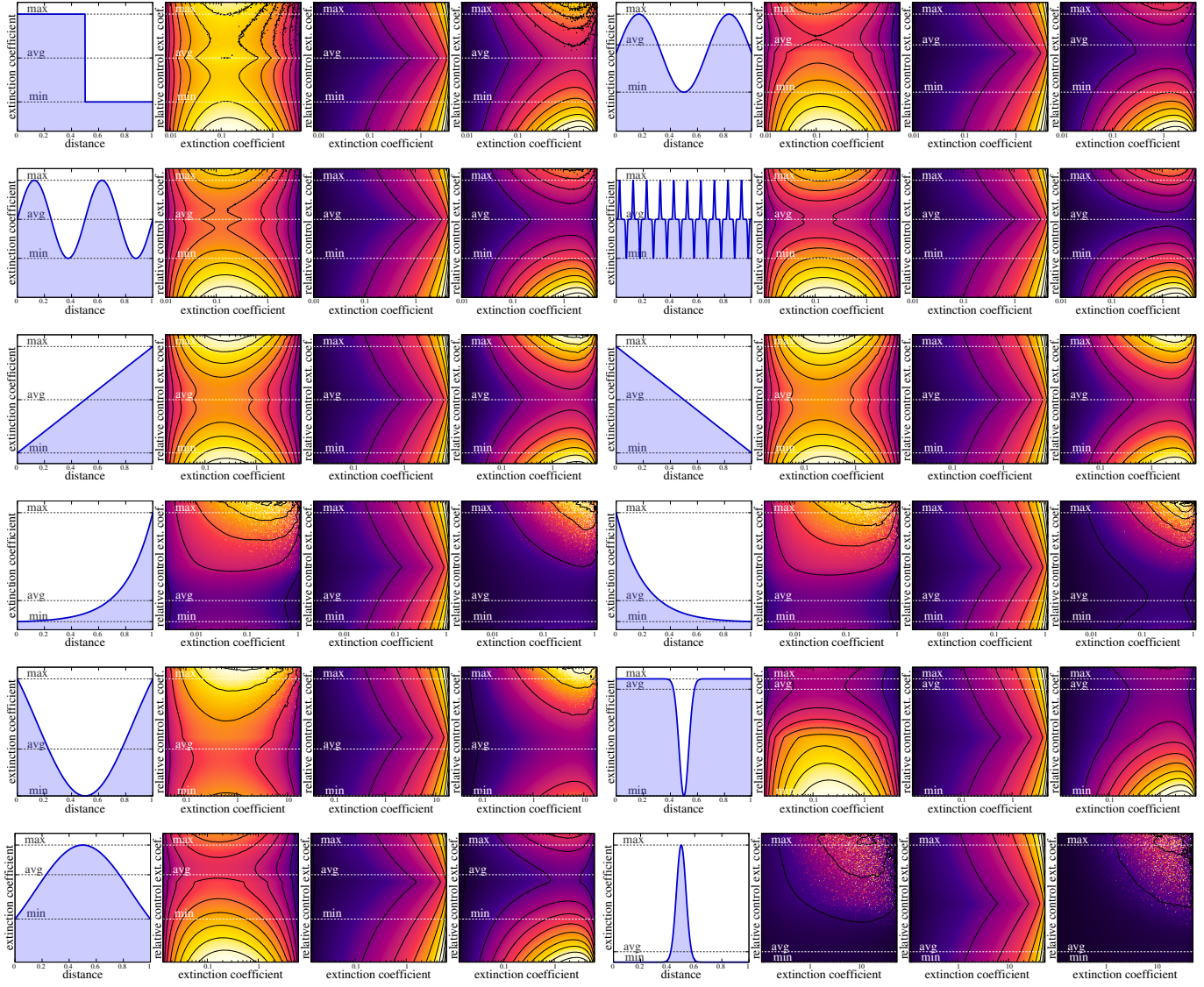
Figure 2: *Comparison of residual tracking with different control extinction coefficients. In* (**b**) *and* (**c**), *we analytically compute the control transmittance (purple) based on the minimum* $\mu(x)$ *along the ray and then apply delta tracking* (**b**) *and ratio tracking* (**c**) *to numerically solve the transmittance (blue curves - individual trackings, black curve - average) through the residual medium. The product of the control and the residual transmittance is represented by the green curve. Ratio tracking can be used with arbitrary control extinctions: in* (**d**) *and* (**e**), *we show examples with the average and maximum* $\mu(x)$ *used as the control extinction.*

Figure 3: *Variance, number of evaluations of the extinction coefficient, and their corresponding product for different extinction functions. Note that independently of the optical thickness (horizontal axis), the product of the variance and cost is minimized quite well by the average extinction coefficient (vertical axis).*
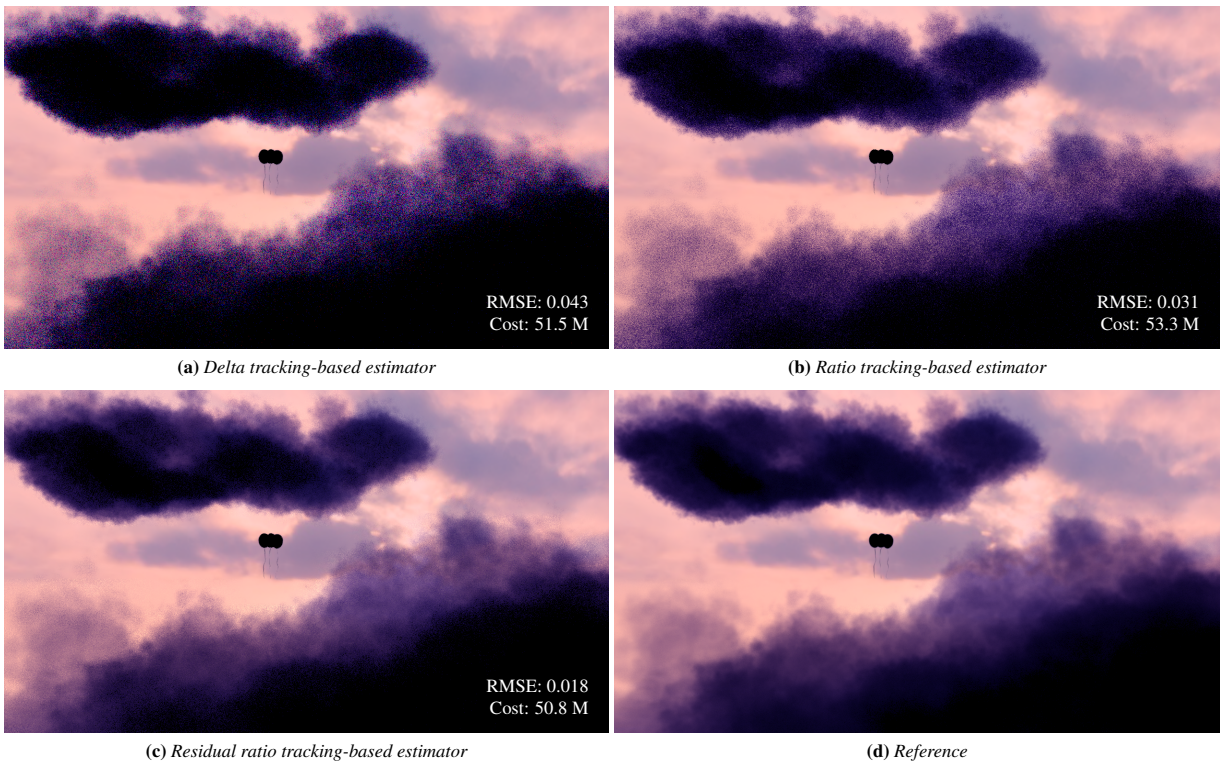
(a) *Delta tracking-based estimator*

(b) *Ratio tracking-based estimator*

(c) *Residual ratio tracking-based estimator*

(d) *Reference*

Figure 4: *An equal-cost comparison of different transmittance estimators used for evaluating transmittance. The images were computed using primary rays only, hence there is no scattering in the medium. The cost of rendering each image is reported as the number of evaluations of the extinction function; each estimator used about 50 million evaluations.*
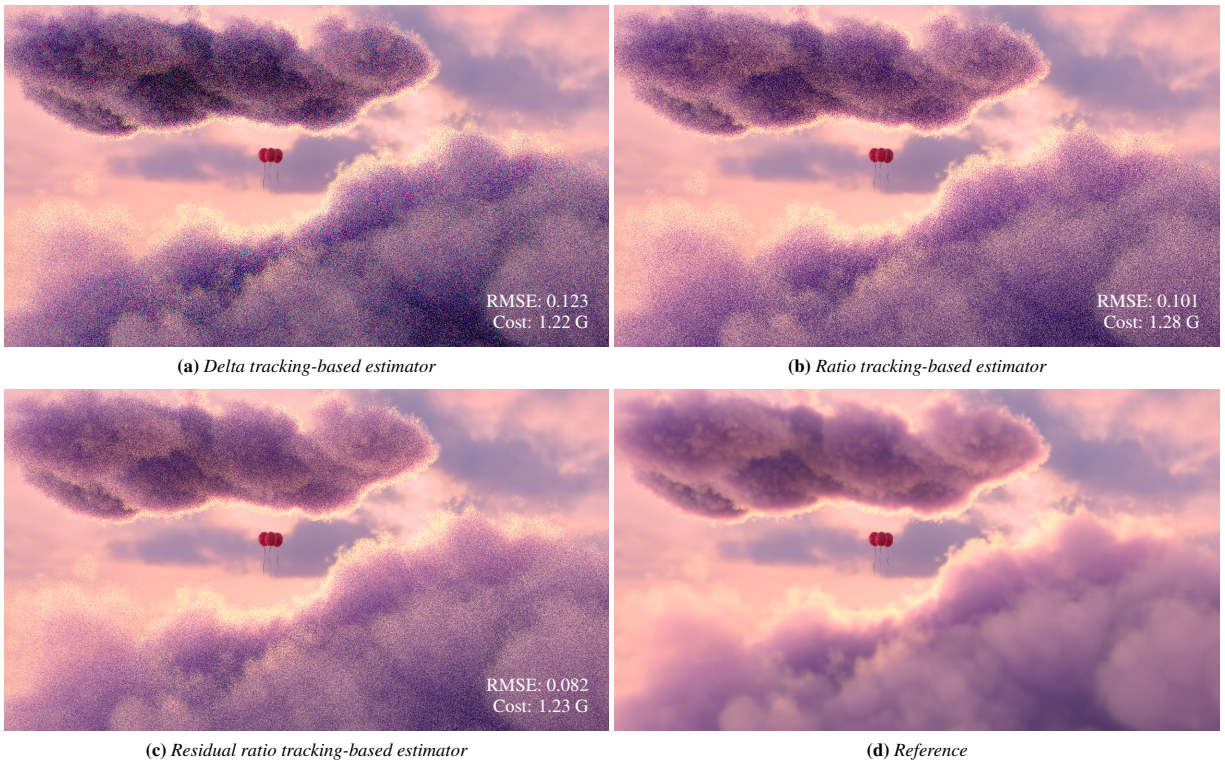

(a) *Delta tracking-based estimator*

(b) *Ratio tracking-based estimator*

(c) *Residual ratio tracking-based estimator*

(d) *Reference*

Figure 5: *A roughly equal-cost comparison of different transmittance estimators used for evaluating transmittance. The images show dual scattering. The cost of rendering each image is reported as the number of evaluations of the extinction function; each estimator used about 1 billion evaluations.*
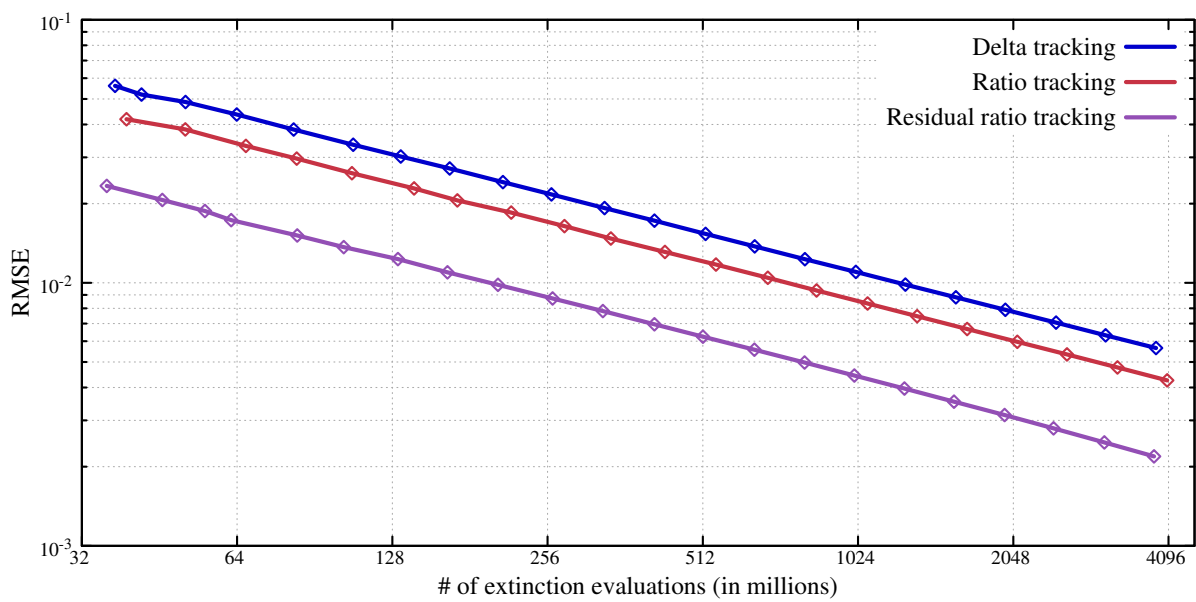
Figure 6: *A log-log convergence plot showing the RMSE for different numbers of evaluations of the extinction function. The data was obtained by progressively rendering the Cloud scene in Figure 4. For the same RMSE, the residual ratio tracking requires about 6× fewer $\mu(x)$ evaluations than delta tracking in this scene.*